FRIEDRICH-SCHILLER-UNIVERSITÄT JENA

Department of Mathematics Dr. Alexander Fromm
Stochastic Analysis (SS 2019)

Exercise sheet 9

Submission: 11.06.2019

Problem 1 - Itô formula (4 Points)

Let $(B_s)_{s\geq 0}$ be a Brownian motion. Use Itô's formula to rewrite the following processes without using stochastic integrals:

(a) $\int_0^t B_s^2 \, dB_s$

(b) $\int_0^t s \, dB_s$

(c) $\int_0^t (B_s + 2s)^2 \, dB_s$

Problem 2 (4 Points)

(a) Let $(B_t)_{t\geq 0}$ be a Brownian motion. Use Itô's formula to show that

(i) $X_t = \frac{B_t}{1+t}$ satisfies for $t \geq 0$

$$X_t = \int_0^t \frac{1}{1+s} \, dB_s - \int_0^t \frac{1}{1+s} X_s \, ds.$$

(ii) $X_t = \sin(B_t)$ satisfies

$$X_t = \int_0^t \sqrt{1 - X_s^2} \, dB_s - \frac{1}{2} \int_0^t X_s \, ds$$

for all $t \leq \tau$, where $\tau = \inf \left\{ s \geq 0 \,|\, B_s \notin [-\frac{\pi}{2}, \frac{\pi}{2}] \right\}$.

(b) Let $(B_t)_{t\geq 0}$ be a Brownian motion. Show that the process

$$X_t = (B_t + t) \exp\left(-B_t - \frac{t}{2} \right), \; t \in \mathbb{R}_{\geq 0},$$

is a martingale.

Problem 3 - Proof of Lemma 10.1 **(7 Points)**

Let $(M_t)_{t\geq 0}$ be a continuous martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ and assume that M is bounded by some constant $C > 0$. Consider a fixed $t \in \mathbb{R}_{\geq 0}$.

(a) Show that for any $\Delta = (t_i)_{0\leq i\leq n} \in \mathcal{P}[0,t]$ and any $k \in \{0,\dots,n\}$ we have

$$\mathbb{E}\left[\sum_{i=k+1}^{n} \left(M_{t_i} - M_{t_{i-1}}\right)^2 \middle| \mathcal{F}_{t_k}\right] = \mathbb{E}\left[(M_t - M_{t_k})^2 \middle| \mathcal{F}_{t_k}\right].$$

(b) Use (a) to show that

$$\mathbb{E}\left[\left(Q_t^{\Delta}(M)\right)^2\right] \leq 12C^2\mathbb{E}\left[(M_t - M_0)^2\right],$$

for all $\Delta \in \mathcal{P}[0,t]$.

(c) Using (b) show that

$$\lim_{\substack{(t_i)_{0\leq i\leq n}=\Delta\in\mathcal{P}[0,t] \\ |\Delta|\to 0}} \mathbb{E}\left[\sum_{i=1}^{n} \left(M_{t_i} - M_{t_{i-1}}\right)^4\right] = 0.$$

Total: 15 Points

Terms of submission:

- Solutions can be submitted in groups of at most 2 students.
- Please submit at the beginning of the lecture or until 9:50 a.m. in room 3523, Ernst-Abbe-Platz 2.